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FINDING EQUIVALENT NETWORK FORMU-  
LATIONS FOR CONSTRAINED NETWORK  
PROBLEMS

Darwin Klingman, et al

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13. ABSTRACT  This paper describes a procedure for determining if constrained network problems (i.e., network problems with additional linear constraints) can be transformed into equivalent pure network problems by a linear transformation involving the node constraints and the extra constraints. Our results extend procedures for problems in which the extra constraints consist of bounding certain partial sums of variables.			

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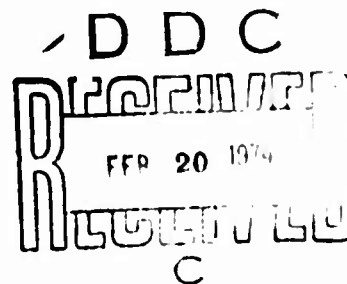
## Constrained Networks

Research Report  
CS 108

FINDING EQUIVALENT NETWORK  
FORMULATIONS FOR CONSTRAINED  
NETWORK PROBLEMS

by

Darwin Klingman  
G. Terry Ross, Jr.



July, 1973



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## 1. INTRODUCTION

In this paper we describe a method for determining if network problems with arbitrary additional linear constraints beyond the standard node constraints can be transformed into pure network problems. These "constrained networks" accurately model numerous resource allocation problems with objectives or restrictions that are not reflected in the node constraints of pure networks. When these seemingly more general linear programming problems can be transformed, they can be solved using any specialized network algorithm. Thus, significant computational savings can be realized over general purpose methods. Our procedure determines if by a linear transformation an "extra" constraint can be transformed into an equivalent bounded sum of variables associated with arcs directed into or away from a single node. If this is possible the procedure finds the linear transformation that yields the equivalent constraint. Finding the appropriate transformation by our method is akin to finding values for dual evaluators of a basic solution to a network problem and lends itself to efficient computer implementation [4]. Further we show how the equivalent bounded sums can be incorporated into an enlarged pure network in a manner analogous to that given by Wagner [7], Manne [2, p. 382] and Charnes [1].

In section 2 we present the procedure for finding equivalent bounded sums of variables. In section 3 we show how these sums or variables can be embodied in the node constraints of an enlarged network. A typical example of the constrained network is given in section 4.

## 2. FINDING EQUIVALENT BOUNDED SUMS OF VARIABLES

The constrained network optimization problem can be formulated mathematically as follows:

minimize:  $\sum_{(i,j) \in A} c_{ij} x_{ij}$

subject to:

$$(1) \sum_{(j,i) \in A} x_{ji} - \sum_{(i,j) \in A} x_{ij} = g_i \text{ for all } i \in N$$

$$(2) \sum_{(i,j) \in A} p_{ij} x_{ij} \leq d$$

$$(3) x_{ij} \geq 0 \text{ for all } (i,j) \in A$$

where  $N$  is a set of nodes or junction points,  $A$  is a set of directed arcs or links between nodes in  $N$ ,  $c_{ij}$  is the unit cost associated with arc  $(i,j)$ ,  $g_k < 0$  ( $>0$ ) is the supply (demand) associated with node  $k \in N$  and  $\sum_{k \in N} g_k = 0$ .

The node conservation constraints (1) and the non-negativity restrictions (3) constitute the standard constraints of the pure network problem. The additional constraint (2) precludes solving this problem with any existing network algorithm. However, if a constraint equivalent to (2) can be found which in turn may be represented by node conservation constraints, the problem can be solved using a network algorithm. The procedure we shall describe determines if there exists a linear combination of the node constraints (1) which when subtracted from the extra constraint (2) yields a bounded sum of variables associated with arcs directed into (or away from) a single node. Such bounded sums will be shown to have equivalent formulations as node conservation constraints in section 3.

To find such a linear combination, we associate a multiplier  $w_j$  with each node constraint and try to determine values for these  $w_j$  such that  $p_{ij} = w_j - w_i$  for every arc  $(i,j) \in A$  whose associated variable  $x_{ij}$  does not appear in the equivalent constraint. It may be observed that if such a linear combination exists, then some linear combination can be found in which the multiplier associated with any single node constraint is assigned a value of zero. This follows from the fact that the coefficient matrix for a network does not have full row rank.

We shall use the following notation in describing our procedure:

$T_k = \{(k,j) \in A\}$  = the set of arcs in  $A$  directed away from node  $k \in N$ .

$H_k = \{(k,j) \in A\}$  = the set of arcs in  $A$  directed into node  $k \in N$ .

The procedure consists of the following three steps:

Step 1. Arbitrarily select a node  $k \in N$  and make the assumption that the extra constraint (2) is equivalent by a linear transformation to a partial sum of variables associated with a subset of arcs  $S$  directed away from a single node (directed into a single node) i.e., we assume constraint (2) is equivalent to

$$(4) \quad \sum_{(i,j) \in S} x_{ij} \leq f \quad \text{where } S \subseteq T_k \text{ (} S \subseteq H_k \text{)}.$$

Step 2. Test the assumption by attempting to find values for all  $w_j, j \in N$  such that  $p_{ij} = w_j - w_i$  for all arcs  $(i,j) \in A-S$  (i.e., try to find a linear combination of the node constraints to subtract from the extra constraint yielding the constraint (4)). An explicit procedure for making this test will be given below.

Step 3. Apply step 1 and step 2 to every subset  $T_k$  and  $H_k$  for  $k \in N$  until the test step verifies one of the following conditions:

- a) the assumption is correct for some  $T_k$  or  $H_k$ .
- b) the assumption has failed for all  $T_k$  and  $H_k$ .
- c) the extra constraint is found to be redundant.
- d) an equivalent constraint reveals that the original problem lacks a feasible solution.

To execute step 2 for node  $q \in N$  and set  $T_q$ , we begin by setting the multiplier  $w_q$  equal to zero. Next we assign values to those multipliers associated with nodes linked to node  $q$  by arcs in  $A-T_q$  (i.e., ignoring the arcs in  $T_q$ ). For example, if an arc  $(i,q) \in A-T_q$  exists we set  $w_i$ , the multiplier for node  $i$  linked to node  $q$  by the arc  $(i,q)$ , using the equation  $p_{iq} = w_q - w_i$  (or equivalently  $w_i = -p_{iq}$ ). The equation  $p_{ij} = w_j - w_i$  must be satisfied for all arcs  $(i,j) \in A-T_q$  for the assumption to be correct, and thus

we check to see if it is satisfied for every arc linking two nodes with assigned multipliers. If this is not the case, we proceed to step 3. Otherwise, we determine values for those unassigned multipliers  $w_i$  associated with nodes linked (by an arc  $(i,j)$  in  $A-T_q$ ) to any node with an assigned multiplier using the equation  $w_j - w_i = p_{ij}$ .

At some point either all multipliers have been assigned values satisfying  $p_{ij} = w_j - w_i$  for all  $(i,j) \in A-T_q$  or the nodes with unassigned multipliers are not connected by arcs in  $A-T_q$  to nodes with assigned multipliers. In the first case, the differences  $p_{qj} - w_j$  are checked for each arc  $(q,j) \in T_q$  to see if they assume one or more than one nonzero value. If all of the differences are equal to zero, then the extra constraint is either redundant or the problem is infeasible (depending on the value of the right hand side of the equivalent constraint since it has the form  $0 \leq d - \sum_{k \in N} w_k g_n$ ).

If all of the differences equal one nonzero value, then the assumption is true and  $S$  is determined by reference to the arcs associated with these nonzero values. If the differences assume more than one distinct nonzero value, then the assumption is false and we return to step 3.

If the nodes with unassigned multipliers are not connected to the nodes with assigned multipliers and the deletion of the arcs in  $T_q$  has created a disconnected network, we arbitrarily select an unassigned multiplier and assign it a variable value  $\theta_1$ . We then proceed as before assigning multiplier values to those  $w_i$  associated with nodes linked to nodes with assigned multipliers. At some point all multipliers have been assigned a value or another disjoint subnetwork has been found. If another disjoint subnetwork exists, we assign some unassigned multiplier a value of  $\theta_2$  and proceed as before. Ultimately, all multipliers will be assigned a value such that



$p_{ij} = w_j - w_i$  for all arcs  $(i,j) \in A - T_q$ . At this point, we proceed to compute the differences  $p_{qj} - w_j$  for each arc  $(q,j) \in T_q$  and to determine values for the  $\theta_j$ 's. A complete mathematical description of this procedure is given below.

#### TEST PROCEDURE

Let  $F$  be the set of indices of nodes whose multipliers have been assigned a value. The test of the assumption in step 1 for  $q \in N$  and set  $T_q$  is performed as follows.

##### Step 2.1

Initially set  $m=0$ ,  $w_q=0$ , and  $F=\{q\}$ . The variable  $m$  is used to denote the number of disjoint subnetworks created by deleting the arcs in  $T_q$ .

##### Step 2.2a

If  $F=N$  go to step 2.3. Otherwise, select an arc  $(i,j) \in A - T_q$  such that  $i \in F$ ,  $j \notin F$  and go to step 2.2b. If no such arc exists, then select an arc  $(i,j) \in A - T_q$  such that  $i \notin F$ ,  $j \in F$  and go to step 2.2c. If no such arc exists, then go to step 2.2d.

##### Step 2.2b

For the arc selected in 2.2a, set  $w_j = p_{ij} + w_i$ . If  $p_{kj} \neq w_j - w_k$  for  $k \in F$  and  $(k,j) \in A - T_q$  or if  $p_{jk} \neq w_k - w_j$  for  $k \in F$  and  $(j,k) \in A$ , then the assumption is false. Proceed to step 3. If no such  $k$  exists set  $F = F \cup \{j\}$  and begin step 2.2a again.

##### Step 2.2c

For the arc selected in 2.2a, set  $w_i = w_j - p_{ij}$ . If  $p_{ik} \neq w_k - w_i$  for  $k \in F$  and  $(i,k) \in A$  or if  $p_{ki} \neq w_i - w_k$  for  $k \in F$  and  $(k,i) \in A - T_q$ , then the assumption is false. Proceed to step 3. If no such  $k$  exists set  $F = F \cup \{i\}$  and begin step 2.2a again.

### Step 2.2c

Because nodes with assigned multipliers are not linked to nodes with unassigned multipliers, set  $m = m + 1$ . Let  $\theta_m$  be a real valued variable.

Select some  $k \in F$  arbitrarily, set  $w_k = 0$ , set  $F = F \cup \{k\}$  and begin step 2.2a again.

### Step 2.3a

For all  $(q,j) \in T_q$  compute  $\pi_{qj} = p_{qj} - w_j$ .

### Step 2.3b

If  $m=0$  and each  $\pi_{qj}$  equals either 0 or some unique non-zero real number  $\delta$ , then the assumption that the extra constraint is equivalent to (4) is correct.

Further  $S$  has been determined ( $S = \{(q,j) \in T_q : \pi_{qj} = \delta\}$ ), and an equivalent constraint is  $\sum_{(i,j) \in S} \delta x_{ij} \leq d - \sum_{k \in N} w_k g_k = f$ . Unity coefficients on the variables  $x_{ij}$  are obtained by multiplying both sides of the inequality by  $1/\delta$ . (This multiplication may reverse the inequality. In this case,  $S$  should be defined as  $S = T_q - S$  and  $f = g_q - f$  to obtain (4).)

If  $m=0$  and the  $\pi_{qj}$  assumes more than one distinct nonzero value the assumption is false. Proceed to step 3. If  $m=0$  and  $\pi_{qj} = 0$  for all  $(q,j) \in T_q$  and  $f \geq 0$  then the original extra constraint is redundant.

Finally, checking the sign of the coefficients and the sign of the right-hand side (in relation to the direction of the inequality) may indicate that the original problem is infeasible.

### Step 2.3c

If  $m > 0$  it is necessary to compute values for each  $\theta_i$ ,  $i = 1, \dots, m$  such that each  $\pi_{qj}$  equals 0 or  $\delta$  before the assumption can be accepted.

Observe that each  $\pi_{qj}$  can be expressed as  $\pi_{qj} = \alpha_j + \sum_{i=1}^m \beta_{ij} \theta_i$  where  $\alpha_j$  is a real number and  $\beta_{ij}$  equals 0, +1, or -1. We define the sets  $J_0 = \{j : \pi_{qj} = \alpha_j, \beta_{ij} = 0\}$  and  $J_i = \{j : \pi_{qj} = \alpha_j + \beta_{ij} \theta_i, \beta_{ij} \neq 0\}$  for  $i=1, \dots, m$ .

- i) If the  $\alpha_j, j \in J_0$  assume more than one distinct nonzero value the assumption is false, proceed to step 3. If the  $\alpha_j, j \in J_0$  are all zero or  $J_0 = \emptyset$ , go to ii below. If the  $\alpha_j, j \in J_0$  assume a unique nonzero value,  $\delta$ , go to iii below.
- ii) Since all  $\alpha_j, j \in J_0$  equal zero (or  $J_0 = \emptyset$ ), then there are two possibilities. First, there exists an index  $i^*$  such that  $-\alpha_j/\beta_j \neq -\alpha_k/\beta_k$  for any  $j \neq k, j, k \in J_1$ . If so, setting  $\theta_{i^*} = -\alpha_j/\beta_j$ , yields  $\pi_{qj} = 0$  and  $\pi_{qk} \neq 0$ . Thus set  $\delta = \pi_{qk}$  and go to iii below. Second, if no such  $i^*$  can be found then by setting  $\theta_i = -\alpha_j/\beta_j$  for some  $j \in J_1$  and for all  $i$  every  $\pi_{qj}$  will equal zero. Using the reasoning in step 2.3a the problem is either infeasible or the constraint is redundant.
- iii) Since at least one of the  $\pi_{qj}$  is nonzero and equals  $\delta$  then the value for the unassigned  $\theta_i$  can be found in the following manner. If  $\delta = 0$ , set  $\theta_i$  such that  $\alpha_j + \beta_j \theta_i \leq 0$  for all  $j \in J_1$  and  $\alpha_j + \beta_j \theta_i = 0$  for at least one  $j \in J_1$ . If  $\delta > 0$ , set  $\theta_i$  such that  $\alpha_j + \beta_j \theta_i \geq 0$  for all  $j \in J_1$  and  $\alpha_j + \beta_j \theta_i = 0$  for at least one  $j \in J_1$ . If this is not possible for some  $i$  then the assumption is false. Proceed to step 3.

Having determined values for all  $\theta_i, i=1, \dots, m$  every  $\pi_{qj}$  is a real valued constant. The reasoning in step 2.3b can be applied to determine the nature of the equivalent constraint.

**Theorem:** If an extra constraint is equivalent by a linear transformation to a bounded sum of variables associated with arcs directed away from a single node, the stated procedure determines an equivalent bounded sum if and only if one exists.

Proof: We first prove that when the algorithm terminates in step 2.3 an

equivalent bounded sum has been identified. This requires two cases.

Case 1. Assume  $m=0$  and a value for every  $w_j$  has been assigned. In this case the values for the  $w_j$  were determined using steps 2.2b and 2.2c.

The equation  $p_{ij} = w_j - w_i$  is satisfied for all arcs  $(i,j) \in A - T_q$ .

Thus, the only variables which appear in an equivalent constraint are those associated with arcs in  $T_q$ . For every arc  $(q,j) \in T_q$ ,  $\pi_{qj}$  is computed and provided each  $\pi_{qj}$  equals either 0 or  $\delta$ , a sum of variables with unity coefficients is obtained.

Case 2. Assume  $m > 0$  and a value for every  $w_j$  has been assigned. In this case it was not possible to assign a constant to some  $w_j$  using the equation  $p_{ij} = w_j - w_i$  for  $(i,j) \in A - T_q$  and  $i \in F$  or a constant to some  $w_i$  using  $p_{ij} = w_j - w_i$  for  $(i,j) \in A - T_q$  and  $j \in F$ . Consequently, some  $w_k$  for an arbitrary node  $k$  in the  $m^{\text{th}}$  disconnected subnetwork is assigned the variable value  $\theta_m$ .

Values for the node multipliers for the other nodes in this subnetwork connected to node  $k$  by arcs in  $A - T_q$  are then determined by the equation  $p_{ik} = w_k - w_i$  or  $p_{kj} = w_j - w_k$ . Again once all node multipliers have been assigned values (constant or variable) that satisfy those equations only arcs in  $T_q$  can appear in the equivalent constraint. Then a value for each  $\theta_i$  is determined such that  $\pi_{qj} = 0$  for  $j \in T_q - S$  and such that  $\pi_{qj} = \delta$  for  $j \in S$ . These  $\theta_i$  must exist for the original constraint to be equivalent to (4).

We next prove that the algorithm will find the equivalent bounded sum when it exists. Assume there exists a linear combination of the node constraints which when subtracted from the extra constraint yields an equivalent constraint bounding the sum of flows on arcs directed away from a single node  $q$ . We note that the rank of the node constraint matrix is  $n-1$  where  $n$  is the number of nodes. By fixing one node multiplier to a particular value, the remaining node multipliers in any linear combination are uniquely

determined by sequentially solving a system of equations each involving only one unknown variable. That is, when the assumption is made that  $S \leq T_q$ , the multiplier  $w_q$  is set equal to 0, and unique multipliers are determined such that  $p_{ij} = w_j - w_i$  for all  $(i,j) \in A - T_q$ . Through this process, the appropriate linear combination is identified.

The procedure for testing the assumption that  $S \in H_q$  is analagous to that given for testing the assumption that  $S \leq T_q$ . The differences are that the values for the  $w_j$  are determined using the arcs  $(i,j) \in A - H_q$ .

An obvious advantage of this procedure is that the subsets  $T_k$  and  $H_k$  must be examined only once to discover an equivalent bounded sum of variables associated with any subset of the arcs in  $T_k$  or  $H_k$ . Further it is not necessary to determine a value for every node multiplier before an assumption for a particular set can be rejected. That is wherever  $p_{ij} \neq w_j - w_i$  for any arc  $(i,j) \in A - T_q$  (or  $A - H_q$ ) the assumption is rejected irrespective of whether all  $w_j$  have been assigned a value. If the network is not connected by arcs in  $A - T_q$ , the values of  $\pi_{qj}$  for all  $j \in J_0$  should be checked before setting a node multiplier equal to  $\theta_1$ . If some  $\pi_{qj}$  is not equal either to 0 or  $\delta$  the current assumption can be rejected. If the  $\pi_{qj}$  for all  $j \in J_0$  equal only 0 or  $\delta$  then a node in a disconnected network should be assigned a value of  $\theta_1$ . When all nodes in this sub-network have been assigned a value, the  $\pi_{qj}$  for  $j \in J_1$  should be checked. This allows assumptions to be rejected without first computing a constant value for every node multiplier.

We have therefore established a procedure to determine if by a linear transformation it is possible to find a bounded sum of variables equivalent to a given constraint. The procedure can be applied to "less than or equal", "greater than or equal", or equality type constraints.

If the original problem includes several extra constraints, then the procedure can be applied to each one individually. In this case we will require that the transformed constraints involve disjoint sets of variables or nested sets of variables in a single node constraint for the problem to be reformulated as an enlarged network by the procedure given in section 3 below. These restrictions parallel those given by Wagner [7] for transportation problems. Our procedure can also be applied to constrained networks with bounded variables and to constrained generalized networks in which the constraint matrix of the generalized network does not have full row rank. In this case the scaling procedure of [3] must be applied first to obtain an equivalent constrained pure network problem.

### 3. INCORPORATING BOUNDED SUMS INTO NODE CONSTRAINTS

A procedure for extending the transportation model to include a bounded sum of variables in a single node constraint has been given by, Wagner [7], Manne [2, p. 382] and Charnes [1]. We state an analogous procedure for networks to complete the reformulation of constrained networks. Let  $S$  be a set of arcs directed into or away from a single node  $k$ . Any network with an additional restriction of the form  $\sum_{(i,j) \in S} x_{ij} = f$  can be transformed into an enlarged network having two additional nodes and at most two additional arcs by the following procedure (We use the notation defined in section 2.)

Step 1. Set  $N = N \cup \{k', k''\}$ .

Step 2. Set  $g_{k'} = -f$  and set  $g_{k''} = f$

Step 3. a) If  $S \subset T_k$  define  $R = \{(k', j) | (k, j) \in S\}$ . For every  $(k', j) \in R$  set  $c_{k', j} = c_{k, j}$ . Set  $A = A \cup R \cup \{(k, k'')\}$  and set  $c_{k, k''} = 0$ . Go to step 4.

b) If  $S \subset H_k$  define  $R = \{(i, k'') | (i, k) \in S\}$ . For every  $(i, k'') \in R$  set  $c_{i, k''} = c_{i, k}$ . Set  $A = A \cup R \cup \{(k', k)\}$  and set  $c_{k', k} = 0$ . Go to step 4.

- Step 4. a) If the constraint is a "less than or equal" type set  $A = A \cup (k', k')$  and set  $c_{k', k'} = 0$ . Go to step 5.
- b) If the constraint is a "greater than or equal" type set  $A = A \cup (k'', k')$  and set  $c_{k'', k'} = 0$ . Go to step 5.
- c) If the constraint is an equality constraint go to step 5.

Step 5. Set  $A = A - S$

The proof of this procedure follows directly from the construction and is omitted.

#### 4.0 APPLICATION AND EXAMPLE

Numerous models have the structure of a network problem with additional linear constraints. These restrictions may represent secondary objectives or restrictions that are not reflected in either the objective function or the standard node constraints. The following example typifies this class of problems.

Consider a network model for the distribution of a daily newspaper from the printing plant to several surrounding communities. Suppose that the objective is to minimize the cost of making the necessary deliveries with the additional provision that the average time for delivering a paper be no more than 6 hours. It should be clear that without this additional restriction a minimum cost solution might conflict with the important customer service objective of quick delivery. In Figure 1 the unit delivery times are indicated in the semi-circle on the corresponding arc, and the net supply or demand figures in thousands of papers are indicated beside each node.

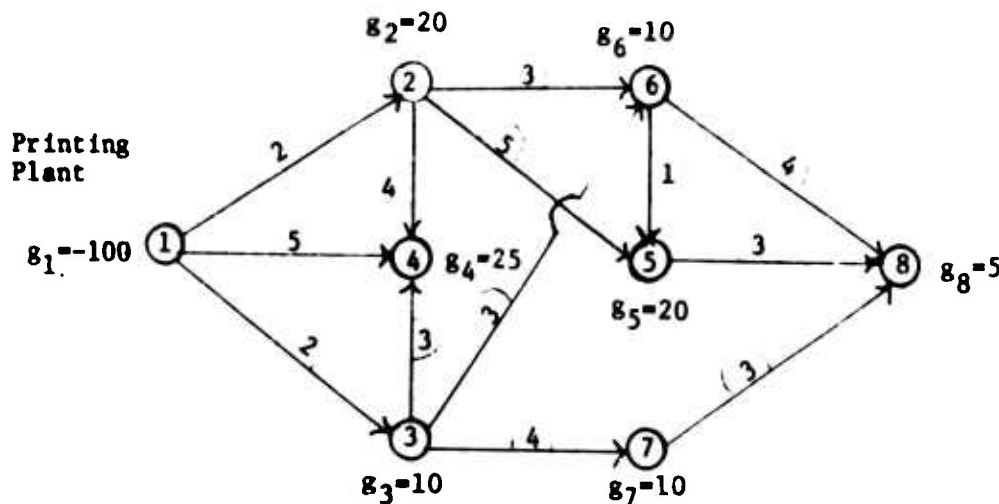


Figure 1.

Thus for this example the total delivery time for 100,000 papers must not exceed 600,000 hours.

Using the procedure given in section 2, we assume first that  $S \leq T_1$ . We set  $m=0$ ,  $w_1=0$  and  $F=\{1\}$ . By step 2.2d we set  $m=m+1$ ,  $w_2 = \theta_1$  and  $F=\{1\} \cup \{2\}$ . Next set  $w_4 = 4 + \theta_1$  and  $F=F \cup \{4\}$ , set  $w_5 = 5 + \theta_1$  and  $F=F \cup \{5\}$ , set  $w_6 = 3 + \theta_1$  and  $F=F \cup \{6\}$ . We set  $w_3 = 1 + \theta_1$  but note that  $p_{36} \neq w_6 - w_3$  and the assumption that  $S \leq T_1$  is rejected. We next assume that  $S \leq T_2$ . Set  $m=0$ ,  $w_2=0$  and  $F=\{2\}$ . By step 2.2c we set  $w_1 = -2$ ,  $F=F \cup \{1\}$ . Next set  $w_4 = 3$ ,  $F=F \cup \{4\}$  and set  $w_3 = 0$ ,  $F=F \cup \{3\}$ . We set  $w_6 = 3$ ,  $F=F \cup \{6\}$  and set  $w_7 = 4$ ,  $F=F \cup \{7\}$ . Next set  $w_5 = 4$ ,  $F=F \cup \{5\}$  and  $w_8 = 7$ ,  $F=F \cup \{8\}$ . At this point  $F=N$  and by step 2.3a we compute  $\pi_{26} = 3 - 3 = 0$ ,  $\pi_{24} = 4 - 3 = 1$ , and  $\pi_{25} = 5 - 4 = 1$ . Hence the original extra constraint is equivalent to the partial sum  $x_{24} + x_{25} \leq 60$ .

Thus the customer service objective can only be satisfied if the total number of papers shipped on routes (2,4) and (2,5) is less than or equal to 60,000. The network in Figure 1 can be transformed into the network in Figure 2 with the partial sum restriction embodied in the node constraints.



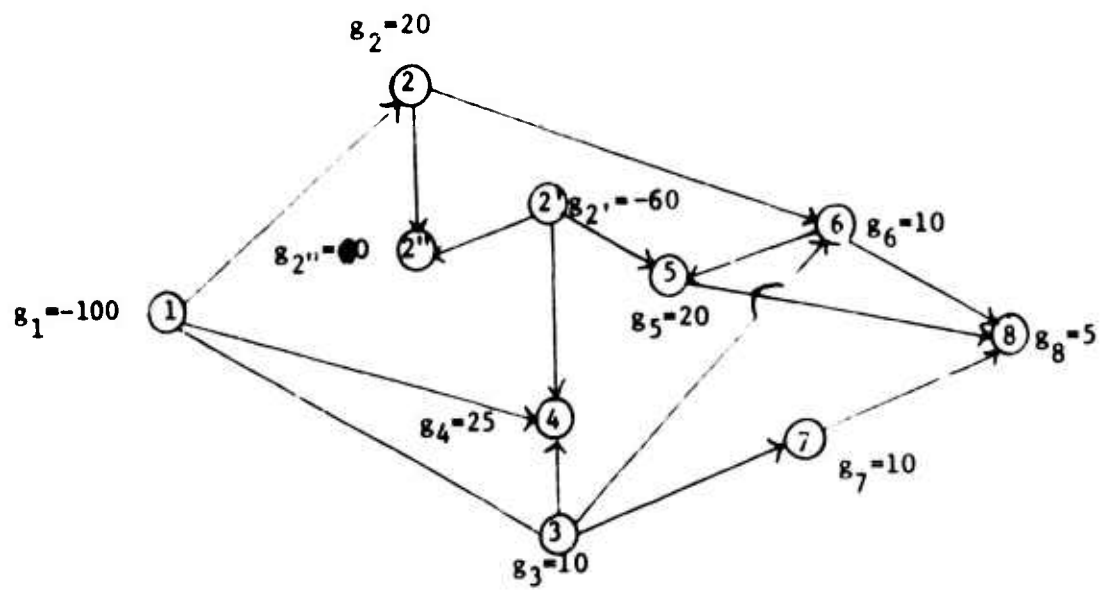


Figure 2.

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